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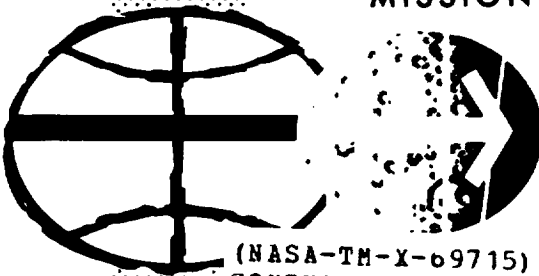
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AN ANALYTIC METHOD FOR
COMPUTING THE STATE TRANSITION
MATRIX WHICH INCLUDES THE
FIRST-ORDER SECULAR PERTURBATIONS
OF PLANET OBLATENESS

Advanced Mission Design Branch

MISSION PLANNING AND ANALYSIS BRANCH



MANNED SPACECRAFT CENTER
HOUSTON, TEXAS

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AN ANALYTIC METHOD FOR COMPUTING THE STATE TRANSITION
MATRIX WHICH INCLUDES THE FIRST-ORDER
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By James C. Kirkpatrick
Advanced Mission Design Branch

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MISSION PLANNING AND ANALYSIS DIVISION
NATIONAL AERONAUTICS AND SPACE ADMINISTRATION
MANNED SPACECRAFT CENTER
HOUSTON, TEXAS

Approved: 
Jack Funk, Chief
Advanced Mission Design Branch

Approved: 
John P. Mayer, Chief
Mission Planning and Analysis Division

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AN ANALYTIC METHOD FOR COMPUTING THE STATE TRANSITION
MATRIX WHICH INCLUDES THE FIRST-ORDER SECULAR
PERTURBATIONS OF PLANET OBLATENESS

By James C. Kirkpatrick

SUMMARY

A method is presented for the analytic computation of the state transition matrix which has the capability of including the first-order secular perturbations of planet oblateness. The method uses the orbital elements in the computations and is formulated solely for elliptic trajectories. The method is not restricted to perturbed or unperturbed trajectories and is not limited in the magnitude of the desired time interval. In addition, the method has the capability of accepting any desired perturbation model provided that analytical derivatives may be obtained.

INTRODUCTION

The state transition matrix is an essential part of space-flight trajectory analysis. In guidance and navigation work it is used to propagate a set of small initial perturbation errors on a reference trajectory to determine their magnitude at some future time. In trajectory optimization work employing primer vector theory, it is used to compute the components of the primer vector and its derivative over the trajectory in question (refs. 1 and 2).

This paper presents a method for computing the state transition matrix which has the capability of including the first-order secular perturbations of planet oblateness. For this purpose, the method performs an initial transformation from Cartesian coordinates to orbital elements and performs all necessary partial derivative computations in terms of orbital elements. The transition matrix in terms of Cartesian coordinates is then formed as the product of three separate matrices.

The formulation employed in the analytical formulation is solely for elliptic trajectory analysis. However, the method is not limited to perturbed or unperturbed orbit computations and is not restricted in the magnitude of the propagation time interval. The approach to the solution lends itself to include short and long period perturbations as well as second-order and third-body effects.

The analytic formulation was initially developed to produce a supplementary matrix which could be added to the transition matrix produced by the program PERTRB (ref. 4) to provide the capability of including the first-order secular perturbations of planet oblateness. As a result, a formulation was developed which would permit the partitioning of the resultant matrix into its unperturbed and perturbation components. After the solution of the problem was completed, it was found that the program PERTRB is limited for elliptic trajectories to time intervals of less than or equal to one period of the orbit. As a result, the initial formulation (which possessed inherent inaccuracies due to the choice of the time of periapsis passage as one of the independent variables) was modified to include the mean anomaly as an independent variable to increase accuracy.

The author gratefully acknowledges the contributions of George H. Born and Claude E. Hildebrand, Jr. to the formulation of the problem and of Ellis W. Henry who provided a number of computer programs used in the evaluation of the results as well as all the test cases used in this study.

ANALYSIS

Definition and Properties of the State Transition Matrix

It is shown in reference 3 that first-order perturbations in position ($\delta \bar{r}$) and velocity ($\delta \bar{V}$) at time t are related to first-order perturbations $\delta \bar{r}_0$ and $\delta \bar{V}_0$ at an initial time t_0 by the following equation

$$\begin{pmatrix} \delta \bar{r} \\ \delta \bar{V} \end{pmatrix} = \Phi(t, t_0) \begin{pmatrix} \delta \bar{r}_0 \\ \delta \bar{V}_0 \end{pmatrix} \quad (1)$$

where $\Phi(t, t_0)$ is the 6×6 transition matrix defined as

$$\frac{d}{dt} [\Phi(t, t_0)] = F(t)\Phi(t, t_0) \quad (2)$$

subject to the conditions

$$\Phi(t_0, t_0) = \Phi(t, t) = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \quad (3)$$

where I is a 3×3 identity matrix and 0 is a 3×3 null matrix. The quantity $F(t)$ is defined by the relation

$$F(t) = \begin{pmatrix} 0 & I \\ G(t) & 0 \end{pmatrix} \quad (4)$$

where $G(t)$ is the gravity gradient matrix defined as

$$G(t) = \begin{pmatrix} \frac{\partial \bar{g}}{\partial \bar{r}} \end{pmatrix} \quad (5)$$

where \bar{g} is the gravitational acceleration vector and \bar{r} is the position vector.

If the equations of motion of a particle considered as a point mass can be derived from a time invariant potential, then the force field may be classified as conservative and the gravity gradient matrix is always symmetric. This means that if U is a time invariant potential function, then

$$\ddot{\bar{r}} = \nabla U = \bar{g} \quad (6)$$

$$G(t) = \left(\frac{\partial \bar{g}}{\partial \bar{r}} \right) = \begin{bmatrix} \frac{\partial^2 U}{\partial x^2} & \frac{\partial^2 U}{\partial x \partial y} & \frac{\partial^2 U}{\partial x \partial z} \\ \frac{\partial^2 U}{\partial y \partial x} & \frac{\partial^2 U}{\partial y^2} & \frac{\partial^2 U}{\partial y \partial z} \\ \frac{\partial^2 U}{\partial z \partial x} & \frac{\partial^2 U}{\partial z \partial y} & \frac{\partial^2 U}{\partial z^2} \end{bmatrix} \quad (7)$$

Battin (ref. 3) derives the gravity gradient matrix for an inverse square force of attraction. It will be shown that the gravity gradient matrix is symmetric when derived from a potential function which includes the second harmonic of the gravitational potential

$$U = \left(\frac{\mu}{r} \right) \left[1 + J \left(\frac{R}{r} \right)^2 \left(\frac{1}{3} - \sin^2 \phi \right) \right] \quad (8)$$

where J is the coefficient of the second harmonic of the potential function, R is the equatorial radius of the attracting body, r is the magnitude of the radial distance from the center of the attracting body to the particle in orbit, μ is the gravitational parameter of the attracting body, and ϕ is the latitude of the particle in orbit measured from the center of the coordinate system fixed at the center of the attracting body. The gravity gradient matrix can be formed by direct differentiation of equation(8) after replacing $\sin \phi$ by z/r .

$$\begin{aligned} \frac{\partial U}{\partial x} &= \mu \left[-\frac{x}{r^3} + JR^2 \left(\frac{-x^3 - xy^2 + 4xz^2}{r^7} \right) \right] \\ \frac{\partial U}{\partial y} &= \mu \left[-\frac{y}{r^3} + JR^2 \left(\frac{-y^3 - yx^2 + 4yz^2}{r^7} \right) \right] \\ \frac{\partial U}{\partial z} &= \mu \left[-\frac{z}{r^3} + JR^2 \left(\frac{-3x^2z - 3y^2z + 2z^3}{r^7} \right) \right] \end{aligned} \quad (9)$$

The second-order partials follow directly from differentiation of the first-order partials of equation (9)

$$\begin{aligned}\frac{\partial^2 U}{\partial x \partial y} &= \frac{\partial^2 U}{\partial y \partial x} = \mu \left[\frac{2xy}{r^5} + JR^2 \left(\frac{5x^3y + 5xy^3 - 30xyz^2}{r^9} \right) \right] \\ \frac{\partial^2 U}{\partial x \partial z} &= \frac{\partial^2 U}{\partial z \partial x} = \mu \left[\frac{2xz}{r^5} + JR^2 \left(\frac{15x^3z + 15xy^2z - 20xz^2}{r^9} \right) \right] \\ \frac{\partial^2 U}{\partial y \partial z} &= \frac{\partial^2 U}{\partial z \partial y} = \mu \left[\frac{2yz}{r^5} + JR^2 \left(\frac{15x^2yz + 15y^5z - 20yz^3}{r^9} \right) \right]\end{aligned}\quad (10)$$

Comparison of the results of equation (10) with the definition of the gravity gradient matrix given by equation (7) shows that the gravity gradient matrix is symmetric.

The importance of the symmetric property of the gravity gradient matrix lies in establishing the symplectic property of the state transition matrix. A symplectic matrix is any even dimensional matrix ϕ which satisfies the condition

$$\phi^T A \phi = A \quad (11)$$

where

$$A = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \quad (12)$$

For any matrix ϕ which satisfies equation (11), the first derivative with respect to time of equation (11) gives

$$\frac{d}{dt} (\phi^T A \phi) = 0 \quad \text{or} \quad \frac{d}{dt} \left[\phi(t, t_0)^T A \phi(t, t_0) \right] = 0 \quad (13)$$

if ϕ is the state transition matrix $\phi(t, t_0)$. Expanding the left hand side of equation (13) and introducing the results of equation (2) gives

$$\begin{aligned} \frac{d}{dt} [\phi(t, t_0)^T A \phi(t, t_0)] &= 0 = \phi(t, t_0)^T F(t)^T A \phi(t, t_0) + \phi(t, t_0)^T A F(t) \phi(t, t_0) \\ &= \phi(t, t_0)^T \begin{bmatrix} -G(t)^T & 0 \\ 0 & I \end{bmatrix} \phi(t, t_0) + \phi(t, t_0)^T \begin{bmatrix} G(t) & 0 \\ 0 & -I \end{bmatrix} \phi(t, t_0) \end{aligned} \quad (14)$$

Equation (14) can hold only if $G(t) = G(t)^T$ which is always true if $G(t)$ is derived from a time invariant potential function such as given by equation (8).

The symplectic property of a state transition matrix is of primary importance in taking its inverse analytically (ref. 3). However, a transition matrix which takes into account only the first-order secular effects of the second harmonic term will not necessarily be symplectic. The reason for this is that the equations of motion which would be required to describe the motion of a particle moving under the effects of a gravitational force which produces only secular effects could not be integrated to produce the same form of the potential given by equation (8). This point was verified in the results of this analysis.

In this study the J term of equation (8) was replaced by $\frac{3}{2} J_{20}$ or simply $\frac{3}{2} J_2$ as only zonal harmonics were considered.

Analytical Computation of the State Transition Matrix

The state transition matrix is computed analytically as the product of three 6×6 matrices in the following manner:

$$\phi(t, t_0) = \left(\frac{\partial S_i(t)}{\partial S_j(t_0)} \right) = \left(\frac{\partial S_i(t)}{\partial \epsilon_j(t)} \right) \left(\frac{\partial \epsilon_i(t)}{\partial \epsilon_j(t_0)} \right) \left(\frac{\partial \epsilon_i(t_0)}{\partial S_j(t_0)} \right) \quad (15)$$

where $i, j = 1, 2, \dots, 6$. These matrices may be defined as follows:

$$\left(\frac{\partial S_i(t)}{\partial S_j(t_0)} \right) = \begin{bmatrix} \frac{\partial x(t)}{\partial x(t_0)} & \frac{\partial x(t)}{\partial y(t_0)} & \dots & \dots & \frac{\partial x(t)}{\partial \dot{z}(t_0)} \\ \frac{\partial y(t)}{\partial x(t_0)} & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \frac{\partial \dot{z}(t)}{\partial x(t_0)} & \dots & \dots & \dots & \frac{\partial \dot{z}(t)}{\partial \dot{z}(t_0)} \end{bmatrix} \quad (16)$$

$$\left(\frac{\partial S_i(t)}{\partial \epsilon_j(t)} \right) = \begin{bmatrix} \frac{\partial x(t)}{\partial a(t)} & \frac{\partial x(t)}{\partial e(t)} & \dots & \dots & \frac{\partial x(t)}{\partial M(t)} \\ \frac{\partial y(t)}{\partial a(t)} & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \frac{\partial \dot{z}(t)}{\partial a(t)} & \dots & \dots & \dots & \frac{\partial \dot{z}(t)}{\partial M(t)} \end{bmatrix} \quad (17)$$

$$\begin{pmatrix} \frac{\partial \epsilon_i(t)}{\partial \epsilon_j(t_0)} \end{pmatrix} = \begin{bmatrix} \frac{\partial a(t)}{\partial a(t_0)} & \frac{\partial a(t)}{\partial e(t_0)} & \dots & \dots & \frac{\partial a(t)}{\partial M(t_0)} \\ \frac{\partial e(t)}{\partial a(t_0)} & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \frac{\partial M(t)}{\partial a(t_0)} & \dots & \dots & \dots & \frac{\partial M(t)}{\partial M(t_0)} \end{bmatrix}$$

(18)

$$\begin{pmatrix} \frac{\partial \epsilon_i(t_0)}{\partial S_j(t_0)} \end{pmatrix} = \begin{bmatrix} \frac{\partial a(t_0)}{\partial x(t_0)} & \frac{\partial a(t_0)}{\partial y(t_0)} & \dots & \dots & \frac{\partial a(t_0)}{\partial \dot{z}(t_0)} \\ \frac{\partial e(t_0)}{\partial x(t_0)} & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \frac{\partial M(t_0)}{\partial x(t_0)} & \dots & \dots & \dots & \frac{\partial M(t_0)}{\partial \dot{z}(t_0)} \end{bmatrix}$$

(19)

In the above expressions, $S_i(t_0)$ and $S_i(t)$ represent the components of the initial and final state vectors referred to the inertial coordinate system. As i increases from 1 to 6, S takes on the representation of $x, y, z, \dot{x}, \dot{y},$ and \dot{z} , respectively. In the same manner, $\epsilon_i(t_0)$ and $\epsilon_j(t)$ represent any one of the orbital elements at the initial and final times. As j increases from 1 to 6, ϵ takes on the representation of $a, e, i, \Omega, \omega,$ and M (or t_p if desired). These symbols represent, respectively, the semimajor axis, eccentricity, inclination, longitude of the ascending node, argument of periapsis, and mean anomaly (t_p is the time of periapsis passage).

Referring to figure 1, if $\ell_1, m_1, n_1,$ and ℓ_2, m_2, n_2 are the direction cosines of $S\xi$ and $S\eta$ with respect to axes $S_x, S_y,$ and S_z , then

$$\begin{aligned} x &= \ell_1 \xi + \ell_2 \eta \\ y &= m_1 \xi + m_2 \eta \\ z &= n_1 \xi + n_2 \eta \\ \dot{x} &= \ell_1 \dot{\xi} + \ell_2 \dot{\eta} \\ \dot{y} &= m_1 \dot{\xi} + m_2 \dot{\eta} \\ \dot{z} &= n_1 \dot{\xi} + n_2 \dot{\eta} \end{aligned} \tag{20}$$

From triangles ATN, ABN, and AKN (see fig. 1)

$$\begin{aligned} \ell_1 &= \cos(AT) = \cos \Omega \cos \omega - \sin \Omega \sin \omega \cos i \\ m_1 &= \cos(AB) = \sin \Omega \cos \omega + \cos \Omega \sin \omega \cos i \\ n_1 &= \cos(AK) = \sin \omega \sin i \end{aligned} \tag{21}$$

From triangles DTN, DBN, DKN

$$\begin{aligned} \ell_2 &= \cos(DT) = -\cos \Omega \sin \omega - \sin \Omega \cos \omega \cos i \\ m_2 &= \cos(DB) = -\sin \Omega \sin \omega + \cos \Omega \cos \omega \cos i \\ n_2 &= \cos(DK) = \cos \omega \sin i \end{aligned} \tag{22}$$

Hence, for elliptic orbital motion, the elements of a state vector can be expressed in terms of the orbital elements with the aid of the following transformations.

$$\begin{aligned}
 \xi &= r \cos f = a(\cos E - e) \\
 \eta &= r \sin f = a\sqrt{1 - e^2} \sin E \\
 \dot{\xi} &= -r \sin f \dot{f} = -a \sin E \dot{E} = -\left(\frac{a^2}{r}\right) \left(\frac{\mu}{a^3}\right)^{1/2} \sin E \\
 \dot{\eta} &= r \cos f \dot{f} = a\sqrt{1 - e^2} \cos E \dot{E} = \left(\frac{a^2}{r}\right) \left(\frac{\mu}{a^3}\right)^{1/2} \cos E
 \end{aligned} \tag{23}$$

where r , f , E , and μ are, respectively, the magnitude of the radius vector, true anomaly, eccentric anomaly, and the gravitational parameter of the attracting body. Replacing equations (23) in equation (20) yields, after some simplification

$$\begin{aligned}
 x &= a [(\cos E - e) \ell_1 + \sqrt{1 - e^2} \sin E \ell_2] \\
 y &= a [(\cos E - e) m_1 + \sqrt{1 - e^2} \sin E m_2] \\
 z &= a [(\cos E - e) n_1 + \sqrt{1 - e^2} \sin E n_2] \\
 \dot{x} &= \sqrt{\frac{\mu}{a}} \left[\frac{\sqrt{1 - e^2} \cos E \ell_2 - \sin E \ell_1}{(1 - e \cos E)} \right] \\
 \dot{y} &= \sqrt{\frac{\mu}{a}} \left[\frac{\sqrt{1 - e^2} \cos E m_2 - \sin E m_1}{(1 - e \cos E)} \right] \\
 \dot{z} &= \sqrt{\frac{\mu}{a}} \left[\frac{\sqrt{1 - e^2} \cos E n_2 - \sin E n_1}{(1 - e \cos E)} \right]
 \end{aligned} \tag{24}$$

The partial derivatives of equations (24) with respect to the orbital elements are given in appendix A. They are used to evaluate the matrices given by equations (17) and (19). The matrix given by equation (19) is formed by numerical inversion of the matrix given by equation (17) evaluated at the initial time, t_0 .

In order to effect the transformation from eccentric anomaly to mean anomaly, it is necessary to consider E to be a function of M and e . If the time of periapsis passage t_p is to be considered one of the independent variables, it is necessary to consider M as a function of a and t_p .

To evaluate the matrix given by equation (18), it is necessary to integrate Lagrange's planetary equations (given in appendix B) with a disturbing function which includes only secular terms. This can be done by replacing the term J in equation (8) by $\frac{3}{2} J$ and r and $\sin \phi$ by the equations (see refs. 5 and 6).

$$r = \frac{a(1 - e^2)}{1 + e \cos f} \quad (25)$$

$$\sin \phi = \sin i \sin (f + \omega)$$

The disturbing function F becomes

$$F = \mu \frac{3}{2} \frac{J_2 R^2}{a^3} \left(\frac{a}{r} \right)^3 \left[\frac{1}{3} - \frac{1}{2} \sin^2 i + \frac{1}{2} \sin^2 i \cos^2 (f + \omega) \right] \quad (26)$$

where $F = U - \frac{\mu}{r}$. Differentiation of equation (26) with respect to a , e , i , Ω , ω , and M allows for direct integration of the Lagrange planetary equations. The transformation of the true anomaly f to mean anomaly M (which is a linear function of time in unperturbed motion) is accomplished by the equation

$$\frac{\partial f}{\partial M} = \left(\frac{a}{r} \right)^2 (1 - e^2)^{1/2} \quad (27)$$

This relationship is derived in appendix C. By this transformation the quantities (a/r) and f in the disturbing function F become functions of e and M only and are periodic with respect to M . Therefore, the terms in F which depend neither on M nor on ω are secular; terms depending on ω but not on M are long-period, while those depending solely on M are short-period. Only those terms causing secular variations of the

orbital elements are considered here. In order to sort out such terms, use is made of the fact that short-period perturbations result from the variations of M while the long-period perturbations arise from the secular variations of m . As a result, it is possible to take the mean value of the disturbing function F with respect to M to obtain the long-period perturbations. To obtain the secular perturbations, it is possible to average with respect to M those parts of the disturbing function depending neither on M nor ω .

Examination of equation (26) shows that it gives rise to a secular and a periodic variation. The secular contributions arise from the terms

$$F_1 = \mu \frac{3}{2} \frac{J_2 R^2}{a^3} \left(\frac{a}{r}\right)^3 \left(\frac{1}{3} - \frac{1}{2} \sin^2 i\right) \quad (28)$$

where F_1 represents only the first-order J_2 part of the disturbing function. Since the main interest of this paper lies with the nonperiodic variation of the elements undergoing perturbations, equation (28) may be averaged over a given revolution. To do this, it is necessary to find the average of $(a/r)^3$ as

$$\overline{\left(\frac{a}{r}\right)^3} = \frac{1}{2\pi} \int_0^{2\pi} \left(\frac{a}{r}\right)^3 dM$$

substituting equation (27) gives

$$\overline{\left(\frac{a}{r}\right)^3} = \frac{1}{2\pi} \int_0^{2\pi} \left(\frac{a}{r}\right)^3 \left(\frac{r}{a}\right)^2 \frac{df}{\sqrt{1-e^2}}$$

substituting the first of equations (25)

$$\begin{aligned} \overline{\left(\frac{a}{r}\right)^3} &= \frac{1}{2\pi} \int_0^{2\pi} \frac{1 + e \cos f}{(1 - e^2)^{3/2}} df \\ &= \frac{1}{2\pi} \int_0^{2\pi} (1 - e^2)^{-3/2} df + \frac{1}{2\pi} \int_0^{2\pi} \frac{e \cos f}{(1 - e^2)^{3/2}} df \\ &= \frac{1}{2\pi} (1 - e^2)^{-3/2} (2\pi - 0) + \frac{e}{2\pi} (1 - e^2)^{-3/2} (-\sin f) \Big|_0^{2\pi} \\ &= (1 - e^2)^{-3/2} \end{aligned} \quad (29)$$

Substituting equation (29) in equation (28) gives

$$F_1 = \mu \frac{3}{2} \frac{J_2 R^2}{a^3} (1 - e^2)^{-3/2} \left(\frac{1}{3} - \frac{1}{2} \sin^2 i \right) \quad (30)$$

Equation (30) is used in the integration of the Lagrange planetary equations to give rise to the variations in the orbital elements given below

$$a = a_0$$

$$e = e_0$$

$$i = i_0$$

$$\Omega = \Omega_0 - \frac{3}{2} \frac{J_2 R^2}{p^2} \sqrt{\frac{\mu}{a^3}} (t - t_0)$$

$$\omega = \omega_0 + \frac{3}{2} \frac{J_2 R^2}{p^2} \sqrt{\frac{\mu}{a^3}} \left(2 - \frac{5}{2} \sin^2 i \right) (t - t_0)$$

$$M = M_0 + \sqrt{\frac{\mu}{a^3}} \left[1 + \frac{3}{2} \frac{J_2 R^2}{p^2} \left(1 - \frac{3}{2} \sin^2 i \right) \sqrt{1 - e^2} \right] (t - t_0)$$

$$p = a(1 - e^2)$$

(31)

The partial derivatives of equation (31) with respect to the orbital elements required to form the matrix given by equation (18) is given in appendix D. If the time of periapsis passage is chosen as the sixth independent variable, the equation for M in equations (31) is replaced by

$$t_p = t_{p_0} - \frac{3}{2} \frac{J_2 R^2}{p^2} \left(1 - \frac{3}{2} \sin^2 i \right) \sqrt{1 - e^2} (t - t_0) \quad (32)$$

as derived in appendix B.

In accordance with the recommendations of reference 5, the expressions for Ω and ω given in equation (31) were modified to read

$$\begin{aligned}\Omega &= \Omega_0 - \frac{3}{2} \frac{J_2 R^2}{p^2} \tilde{n} \cos i (t - t_0) \\ \omega &= \omega_0 + \frac{3}{2} \frac{J_2 R^2}{p^2} \tilde{n} \left(2 - \frac{5}{2} \sin^2 i \right) (t - t_0)\end{aligned}\tag{33}$$

where

$$\tilde{n} = \sqrt{\frac{\mu}{a^3}} \left[1 + \frac{3}{2} \frac{J_2 R^2}{p^2} \left(1 - \frac{3}{2} \sin^2 i \right) \sqrt{1 - e^2} \right]\tag{34}$$

THE PROGRAM

The equations for computing the state transition matrix by the analytic method have been programed in double precision using the Fortran V language for the UNIVAC 1108 digital computer. The entire package may be obtained by calling the subroutine STD. Subroutine STD is the driving or control subprogram for the entire package.

The subroutine STD converts the initial state vector and time to nondimensional units and calls the necessary subroutine subprograms which compute the state transition matrix. At the completion of the computations, the elements of the state transition matrix are converted to the units of the state vector in the calling argument list.

STD calls four additional subroutines: STATE, SECULR, MATIN, and MLTMAT. Subroutine STATE calls two additional subprograms; the function DDA and the subroutine MODEL. These subprograms together with the driving subprogram STD may be considered a black box. However, these subprograms perform the following operations.

After converting the initial state vector and time to nondimensional units, STD calls STATE twice in succession. On its initial call, STATE converts the now normalized input state vector to orbital elements and proceeds to compute the matrix given by equation (17) evaluated at the initial time. This matrix is returned to STD and is later inverted by a call to the subroutine MATIN to form the matrix given by equation (19).

On the second call to STATE, the orbital elements are corrected to their value at the end of the time interval in accordance with the equations of the model for the problem. These corrections are brought about by a call from STATE to subroutine MODEL, which contains the equations of the model. After the corrections from MODEL have been returned, a call to the function DDA from STATE corrects mean anomaly at the final time to the eccentric anomaly by a Newton-Raphson iteration on Kepler's equation. With the orbital elements corrected to their values at the final time, STATE proceeds with the computation of the matrix given by equation (17) evaluated at the final time. This matrix is returned to STD which in turn calls SECULR.

Subroutine SECULR computes the matrix given by equation (18). This matrix is returned to STD where it is premultiplied by the matrix given by equation (17) on the first of two successive calls to subroutine MLTMAT. The product of these matrices is postmultiplied by the matrix given by equation (19). This operation is performed on the second call to MLTMAT. The product of these three matrices is the state transition matrix. The elements of the state transition matrix as returned by MLTMAT are converted from nondimensional units to the units of the input state vector and returned as the state transition matrix to the calling program.

A flow diagram of the program for computing the state transition matrix by the analytic method is given in figure 2. The calling statement for the subroutine is: CALL STD (XMU,RO,XJ2,T, \bar{R}_0 , \bar{V}_0 ,TK,RK,VK,PHI).

The arguments in the list are, respectively, the gravitational parameter, equatorial radius, and the coefficient for the second harmonic of the potential function for the attracting body, the time increment, the position and velocity vectors (each an array of 3×1 column vectors), conversion factors to convert the units of the time and the Cartesian coordinates of the position and velocity vectors to the same units of the gravitational parameter and equatorial radius of the attracting body, and the 6×6 array which contains the state transition matrix in the same units as the position and velocity vectors. Only the gravitational parameter and equatorial radius must be supplied in a consistent system of units.

The time required to compute the state transition matrix by the analytic method is approximately 25 milliseconds.

CONCLUSIONS

The state transition matrix can be computed analytically for any specific model desired. The formulation considered in this paper was prepared specifically for elliptic orbit trajectories, and has the capability of extending through any desired number of revolutions.

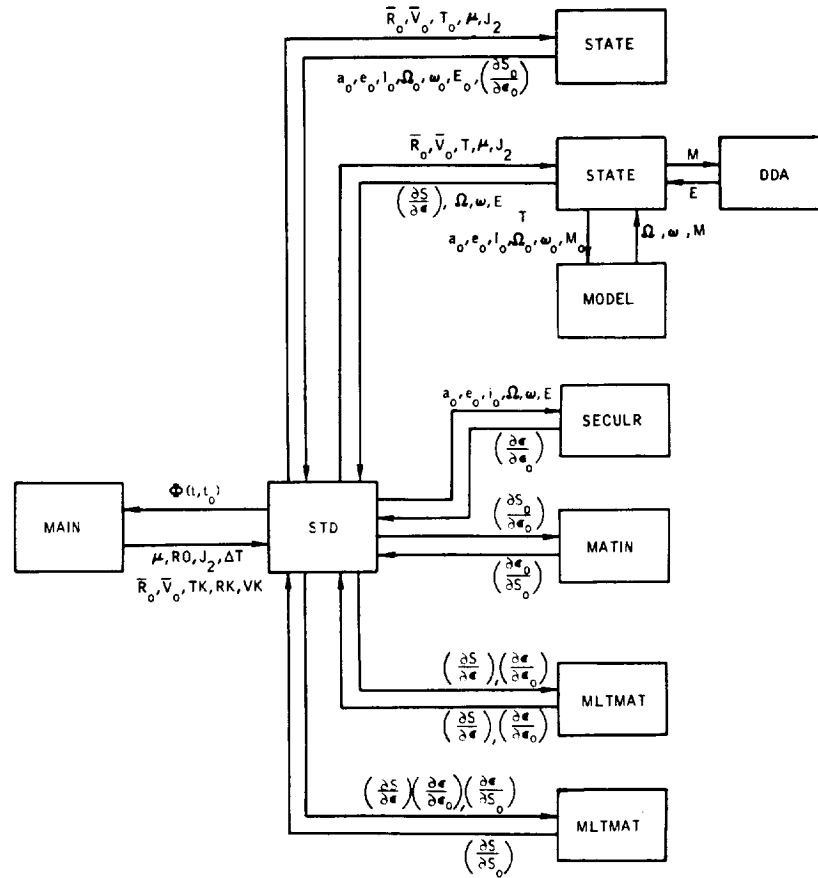


Figure 2. - Flow chart for analytic computation of state transition matrix.

However, any desired type of trajectory can also be considered in the same manner. In addition, any specific perturbation model can also be included. For this paper, a model which considered only the first-order secular perturbations was included. The method has speed and accuracy within the bounds of the specific model considered. This fact was verified by comparison of the results obtained by the analytic method with the results of a numerical technique described in reference 7.

APPENDIX A
COMPUTATION OF PARTIAL DERIVATIVES OF THE CARTESIAN
COORDINATES WITH RESPECT TO THE ORBITAL ELEMENTS

APPENDIX A

COMPUTATION OF PARTIAL DERIVATIVES OF THE CARTESIAN
COORDINATES WITH RESPECT TO THE ORBITAL ELEMENTS

In order to write the partial derivatives of the Cartesian coordinates with respect to the orbital elements, it is convenient to first differentiate the direction cosines.

$$l_1 = \cos \Omega \cos \omega - \sin \Omega \sin \omega \cos i$$

$$(\partial l_1 / \partial i) = \sin \Omega \sin \omega \sin i$$

$$(\partial l_1 / \partial \Omega) = -\sin \Omega \cos \omega - \cos \Omega \sin \omega \cos i$$

$$(\partial l_1 / \partial \omega) = -\cos \Omega \sin \omega - \sin \Omega \cos \omega \cos i$$

$$m_1 = \sin \Omega \cos \omega + \cos \Omega \sin \omega \cos i$$

$$(\partial m_1 / \partial i) = -\cos \Omega \sin \omega \sin i$$

$$(\partial m_1 / \partial \Omega) = \cos \Omega \cos \omega - \sin \Omega \sin \omega \cos i$$

$$(\partial m_1 / \partial \omega) = -\sin \Omega \sin \omega + \cos \Omega \cos \omega \cos i$$

$$n_1 = \sin \omega \sin i$$

$$(\partial n_1 / \partial i) = \sin \omega \cos i$$

$$(\partial n_1 / \partial \Omega) = 0$$

$$(\partial n_1 / \partial \omega) = \cos \omega \sin i$$

$$l_2 = -\cos \Omega \sin \omega - \sin \Omega \cos \omega \cos i$$

$$(\partial l_2 / \partial i) = \sin \Omega \cos \omega \sin i$$

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$$(\partial \ell_2 / \partial \Omega) = \sin \Omega \sin \omega - \cos \Omega \cos \omega \cos i$$

$$(\partial \ell_2 / \partial \omega) = -\cos \Omega \cos \omega + \sin \Omega \sin \omega \cos i$$

$$m_2 = -\sin \Omega \sin \omega + \cos \Omega \cos \omega \cos i$$

$$(\partial m_2 / \partial i) = -\cos \Omega \cos \omega \sin i$$

$$(\partial m_2 / \partial \Omega) = -\cos \Omega \sin \omega - \sin \Omega \cos \omega \cos i$$

$$(\partial m_2 / \partial \omega) = -\sin \Omega \cos \omega - \cos \Omega \sin \omega \cos i$$

$$n_2 = \cos \omega \sin i$$

$$(\partial n_2 / \partial i) = \cos \omega \cos i$$

$$(\partial n_2 / \partial \Omega) = 0$$

$$(\partial n_2 / \partial \omega) = -\sin \omega \sin i$$

The partial derivatives of the Cartesian coordinates with respect to the orbital elements can now be written. However, before this can be done it is necessary to first specify the six independent variables of the transformation.

If $S = S_i(a, e, i, \Omega, \omega, E)$ and $E = E(M, e)$, then $S = S_i(a, e, i, \Omega, \omega, M)$ and $(\partial S / \partial a) = (\partial S_i / \partial a)$, where $i = 1, 2, \dots, 6$, then

$$(\partial S / \partial e) = (\partial S_i / \partial e) + (\partial S / \partial E)(\partial E / \partial e)$$

$$(\partial S / \partial i) = (\partial S_i / \partial i)$$

$$(\partial S / \partial \Omega) = (\partial S_i / \partial \Omega)$$

$$(\partial S / \partial \omega) = (\partial S_i / \partial \omega)$$

$$(\partial S / \partial M) = (\partial S_i / \partial E)(\partial E / \partial M)$$

If $E = E(M, e)$ and $M = M(a, t_p)$ then $E = E[M(a, t_p), e]$ and
 $S = S_i(a, e, i, \Omega, \omega, t_p)$ and

$$(\partial S / \partial a) = (\partial S_i / \partial a) + (\partial S_i / \partial E)(\partial E / \partial M)(\partial M / \partial a)$$

$$(\partial S / \partial e) = (\partial S_i / \partial e) + (\partial S_i / \partial E)(\partial E / \partial e)$$

$$(\partial S / \partial i) = (\partial S_i / \partial i)$$

$$(\partial S / \partial \Omega) = (\partial S_i / \partial \Omega)$$

$$(\partial S / \partial \omega) = (\partial S_i / \partial \omega)$$

$$(\partial S / \partial t_p) = (\partial S_i / \partial E)(\partial E / \partial M)(\partial M / \partial t_p)$$

The additional partial derivatives needed are given as follows:

$$M = \sqrt{\frac{\mu}{a^3}} (t - t_p) = E - e \sin E$$

$$(\partial M / \partial a) = \frac{3}{2a} \sqrt{\frac{\mu}{a^3}} (t - t_p)$$

$$(\partial M / \partial e) = (\partial E / \partial e) - \sin E - e \cos E (\partial E / \partial e)$$

but $(\partial M / \partial e) = 0$, therefore

$$(\partial E / \partial e) = \sin E / (1 - e \cos E)$$

$$(\partial M / \partial E) = 1 - e \cos E$$

$$(\partial E / \partial M) = 1 / (1 - e \cos E)$$

The partial derivatives of the Cartesian coordinate with respect to the orbital elements become (for $S = S_i(a, e, i, \Omega, \omega, t_p)$)

$$x = a[(\cos E - e) \ell_1 + a\sqrt{1 - e^2} \sin E \ell_2]$$

$$\begin{aligned} (\partial x / \partial a) &= [\cos E - e - a \sin E (\partial E / \partial M)(\partial M / \partial a)] \ell_1 \\ &\quad + [\sqrt{1 - e^2} \sin E + a\sqrt{1 - e^2} \cos E (\partial E / \partial M)(\partial M / \partial a)] \ell_2 \end{aligned}$$

$$\begin{aligned} (\partial x / \partial e) &= -a[1 + \sin E (\partial E / \partial e)] \ell_1 \\ &\quad + a[\sqrt{1 - e^2} \cos E (\partial E / \partial e) - e \sin E / \sqrt{1 - e^2}] \ell_2 \end{aligned}$$

$$(\partial x / \partial i) = a[(\cos E - e)(\partial \ell_1 / \partial i) + \sqrt{1 - e^2} \sin E (\partial \ell_2 / \partial i)]$$

$$(\partial x / \partial \Omega) = a[(\cos E - e)(\partial \ell_1 / \partial \Omega) + \sqrt{1 - e^2} \sin E (\partial \ell_2 / \partial \Omega)]$$

$$(\partial x / \partial \omega) = a[(\cos E - e)(\partial \ell_1 / \partial \omega) + \sqrt{1 - e^2} \sin E (\partial \ell_2 / \partial \omega)]$$

$$\begin{aligned} (\partial x / \partial t_p) &= (\partial x / \partial M)(\partial M / \partial t_p) = (\partial x / \partial E)(\partial E / \partial M)(\partial M / \partial t_p) \\ &= a[-\sin E \ell_1 + \sqrt{1 - e^2} \cos E \ell_2] (\partial E / \partial M)(\partial M / \partial t_p) \end{aligned}$$

$$y = a[(\cos E - e)m_1 + \sqrt{1 - e^2} \sin E m_2]$$

$$\begin{aligned} (\partial y / \partial a) &= [\cos E - e - a \sin E (\partial E / \partial M)(\partial M / \partial a)]m_1 \\ &\quad + [\sqrt{1 - e^2} \sin E + a\sqrt{1 - e^2} \cos E (\partial E / \partial M)(\partial M / \partial a)]m_2 \end{aligned}$$

$$\begin{aligned} (\partial y / \partial e) &= -a[\sin E (\partial E / \partial e) + 1]m_1 \\ &\quad + a[\sqrt{1 - e^2} \cos E (\partial E / \partial e) - e \sin E / \sqrt{1 - e^2}]m_2 \end{aligned}$$

$$(\partial y / \partial i) = a[(\cos E - e)(\partial m_1 / \partial i) + \sqrt{1 - e^2} \sin E (\partial m_2 / \partial i)]$$

$$(\partial y / \partial \Omega) = a[(\cos E - e)(\partial m_1 / \partial \Omega) + \sqrt{1 - e^2} \sin E (\partial m_2 / \partial \Omega)]$$

$$(\partial y / \partial \omega) = a[(\cos E - e)(\partial m_1 / \partial \omega) + \sqrt{1 - e^2} \sin E (\partial m_2 / \partial \omega)]$$

$$(\partial y / \partial t_p) = a[-\sin E m_1 + \sqrt{1 - e^2} \cos E m_2] (\partial E / \partial M)(\partial M / \partial t_p)$$

$$z = a[(\cos E - e)n_1 + \sqrt{1 - e^2} \sin E n_2]$$

$$\begin{aligned} (\partial z / \partial a) &= [\cos E - e - a \sin E (\partial E / \partial M)(\partial M / \partial a)]n_1 \\ &\quad + [\sqrt{1 - e^2} \sin E + a\sqrt{1 - e^2} \cos E (\partial E / \partial M)(\partial M / \partial a)]n_2 \end{aligned}$$

$$\begin{aligned} (\partial z / \partial e) &= -a[\sin E (\partial E / \partial e) + 1]n_1 \\ &\quad + a[\sqrt{1 - e^2} \cos E (\partial E / \partial e) - e \sin E / \sqrt{1 - e^2}]n_2 \end{aligned}$$

$$(\partial z / \partial i) = a[(\cos E - e)(\partial n_1 / \partial i) + \sqrt{1 - e^2} \sin E (\partial n_2 / \partial i)]$$

$$(\partial z / \partial \Omega) = a[(\cos E - e)(\partial n_1 / \partial \Omega) + \sqrt{1 - e^2} \sin E (\partial n_2 / \partial \Omega)]$$

$$(\partial z / \partial \omega) = a[(\cos E - e)(\partial n_1 / \partial \omega) + \sqrt{1 - e^2} \sin E (\partial n_2 / \partial \omega)]$$

$$(\partial z / \partial t_p) = a[-\sin E n_1 + \sqrt{1 - e^2} \cos E n_2] (\partial E / \partial M)(\partial M / \partial t_p)$$

$$\dot{x} = \sqrt{\frac{\mu}{a}} \left[\frac{\sqrt{1-e^2} \cos E \ell_2 - \sin E \ell_1}{(1-e \cos E)} \right]$$

$$\begin{aligned} \left(\frac{\partial \dot{x}}{\partial a} \right) = & -\sqrt{\frac{\mu}{a}} \left\{ \left[\frac{\sqrt{1-e^2} \sin E \ell_2 - \sin E \ell_1}{(1-e \cos E)} \right] \left[\frac{1}{2a} + \frac{e \sin E}{(1-e \cos E)} \left(\frac{\partial E}{\partial M} \right) \left(\frac{\partial M}{\partial a} \right) \right] \right. \\ & \left. + \left[\frac{\sqrt{1-e^2} \sin E \ell_2 + \cos E \ell_1}{(1-e \cos E)} \right] \left(\frac{\partial E}{\partial M} \right) \left(\frac{\partial M}{\partial a} \right) \right\} \end{aligned}$$

$$\begin{aligned} \left(\frac{\partial \dot{x}}{\partial e} \right) = & \frac{-\sqrt{\mu/a}}{(1-e \cos E)} \left\{ \frac{e \cos E \ell_2}{\sqrt{1-e^2}} + \left[\sqrt{1-e^2} \sin E \ell_2 + \cos E \ell_1 \right] \left(\frac{\partial E}{\partial e} \right) \right. \\ & \left. + \left[\frac{\sqrt{1-e^2} \cos E \ell_2 - \sin E \ell_1}{(1-e \cos E)} \right] \left[e \sin E \left(\frac{\partial E}{\partial e} \right) - \cos E \right] \right\} \end{aligned}$$

$$\left(\frac{\partial \dot{x}}{\partial i} \right) = \frac{\sqrt{\mu/a}}{(1-e \cos E)} \left[\sqrt{1-e^2} \cos E \left(\frac{\partial \ell_2}{\partial i} \right) - \sin E \left(\frac{\partial \ell_1}{\partial i} \right) \right]$$

$$\left(\frac{\partial \dot{x}}{\partial \Omega} \right) = \frac{\sqrt{\mu/a}}{(1-e \cos E)} \left[\sqrt{1-e^2} \cos E \left(\frac{\partial \ell_2}{\partial \Omega} \right) - \sin E \left(\frac{\partial \ell_1}{\partial \Omega} \right) \right]$$

$$\left(\frac{\partial \dot{x}}{\partial \omega} \right) = \frac{\sqrt{\mu/a}}{(1-e \cos E)} \left[\sqrt{1-e^2} \cos E \left(\frac{\partial \ell_2}{\partial \omega} \right) - \sin E \left(\frac{\partial \ell_1}{\partial \omega} \right) \right]$$

$$\left(\frac{\partial \dot{x}}{\partial t_p} \right) = -\sqrt{\frac{\mu}{a}} \left[\frac{\sqrt{1-e^2} \sin E \ell_2 + (\cos E - e) \ell_1}{(1-e \cos E)^2} \right] \left(\frac{\partial E}{\partial M} \right) \left(\frac{\partial M}{\partial t_p} \right)$$

$$\dot{y} = \sqrt{\frac{\mu}{a}} \left[\frac{\sqrt{1-e^2} \cos E m_2 - \sin E m_1}{(1-e \cos E)} \right]$$

$$\left(\frac{\partial \dot{y}}{\partial a}\right) = -\sqrt{\frac{\mu}{a}} \left\{ \left[\frac{\sqrt{1-e^2} \sin E m_2 - \sin E m_1}{(1-e \cos E)} \right] \left[\frac{1}{2a} + \frac{e \sin E}{(1-e \cos E)} \right] \left(\frac{\partial E}{\partial M}\right) \left(\frac{\partial M}{\partial a}\right) \right. \\ \left. + \left[\frac{\sqrt{1-e^2} \sin E m_2 + \cos E m_1}{(1-e \cos E)} \right] \left(\frac{\partial E}{\partial M}\right) \left(\frac{\partial M}{\partial a}\right) \right\}$$

$$\left(\frac{\partial \dot{y}}{\partial e}\right) = \frac{-\sqrt{\mu/a}}{(1-e \cos E)} \left\{ \frac{e \cos E m_2}{\sqrt{1-e^2}} + \left[\frac{\sqrt{1-e^2} \sin E m_2 + \cos E m_1}{\sqrt{1-e^2}} \right] \left(\frac{\partial E}{\partial e}\right) \right. \\ \left. + \left[\frac{\sqrt{1-e^2} \cos E m_2 - \sin E m_1}{(1-e \cos E)} \right] \left[e \sin E \left(\frac{\partial E}{\partial e}\right) - \cos E \right] \right\}$$

$$\left(\frac{\partial \dot{y}}{\partial i}\right) = \frac{\sqrt{\mu/a}}{(1-e \cos E)} \left[\sqrt{1-e^2} \cos E \left(\frac{\partial m_2}{\partial i}\right) - \sin E \left(\frac{\partial m_1}{\partial i}\right) \right]$$

$$\left(\frac{\partial \dot{y}}{\partial \Omega}\right) = \frac{\sqrt{\mu/a}}{(1-e \cos E)} \left[\sqrt{1-e^2} \cos E \left(\frac{\partial m_2}{\partial \Omega}\right) - \sin E \left(\frac{\partial m_1}{\partial \Omega}\right) \right]$$

$$\left(\frac{\partial \dot{y}}{\partial \omega}\right) = \frac{\sqrt{\mu/a}}{(1-e \cos E)} \left[\sqrt{1-e^2} \cos E \left(\frac{\partial m_2}{\partial \omega}\right) - \sin E \left(\frac{\partial m_1}{\partial \omega}\right) \right]$$

$$\left(\frac{\partial \dot{y}}{\partial t_p}\right) = -\sqrt{\frac{\mu}{a}} \left[\frac{\sqrt{1-e^2} \sin E m_2 + (\cos E - e) m_1}{(1-e \cos E)^2} \right] \left(\frac{\partial E}{\partial M}\right) \left(\frac{\partial M}{\partial t_p}\right)$$

$$\dot{z} = \sqrt{\frac{\mu}{a}} \left[\frac{\sqrt{1-e^2} \cos E n_2 - \sin E n_1}{(1-e \cos E)} \right]$$

$$\left(\frac{\partial \dot{z}}{\partial a}\right) = -\sqrt{\frac{\mu}{a}} \left\{ \left[\frac{\sqrt{1-e^2} \cos E n_2 - \sin E n_1}{(1-e \cos E)} \right] \left[\frac{1}{2a} + \frac{e \sin E}{(1-e \cos E)} \right] \left(\frac{\partial E}{\partial M}\right) \left(\frac{\partial M}{\partial a}\right) \right. \\ \left. + \left[\frac{\sqrt{1-e^2} \sin E n_2 + \cos E n_1}{(1-e \cos E)} \right] \left(\frac{\partial E}{\partial M}\right) \left(\frac{\partial M}{\partial a}\right) \right\}$$

$$\begin{aligned} \left(\frac{\partial \dot{z}}{\partial e} \right) &= \frac{-\sqrt{\mu/a}}{(1 - e \cos E)} \left\{ \frac{e \cos E n_2}{\sqrt{1 - e^2}} + \left[\sqrt{1 - e^2} \sin E n_2 + \cos E n_1 \right] \left(\frac{\partial E}{\partial e} \right) \right. \\ &\quad \left. + \left[\frac{\sqrt{1 - e^2} \cos E n_2 - \sin E n_1}{(1 - e \cos E)} \right] \left[e \sin E \left(\frac{\partial E}{\partial e} \right) - \cos E \right] \right\} \end{aligned}$$

$$\left(\frac{\partial \dot{z}}{\partial i} \right) = \frac{\sqrt{\mu/a}}{(1 + e \cos E)} \left[\sqrt{1 - e^2} \cos E \left(\frac{\partial n_2}{\partial i} \right) - \sin E \left(\frac{\partial n_1}{\partial i} \right) \right]$$

$$\left(\frac{\partial \dot{z}}{\partial \Omega} \right) = \frac{\sqrt{\mu/a}}{(1 - e \cos E)} \left[\sqrt{1 - e^2} \cos E \left(\frac{\partial n_2}{\partial \Omega} \right) - \sin E \left(\frac{\partial n_1}{\partial \Omega} \right) \right]$$

$$\left(\frac{\partial \dot{z}}{\partial \omega} \right) = \frac{\sqrt{\mu/a}}{(1 - e \cos E)} \left[\sqrt{1 - e^2} \cos E \left(\frac{\partial n_2}{\partial \omega} \right) - \sin E \left(\frac{\partial n_1}{\partial \omega} \right) \right]$$

$$\left(\frac{\partial \dot{z}}{\partial t_p} \right) = - \sqrt{\frac{\mu}{a}} \left[\frac{\sqrt{1 - e^2} \sin E n_2 + (\cos E - e) n_1}{(1 - e \cos E)^2} \right] \left(\frac{\partial E}{\partial M} \right) \left(\frac{\partial M}{\partial t_p} \right)$$

If $S = S_i(a, e, i, \Omega, \omega, M)$, the following changes must be made:

$$(\partial x / \partial a) = (\cos E - e) \ell_1 + \sqrt{1 - e^2} \sin E \ell_2$$

$$(\partial x / \partial M) = a(-\sin E \ell_1 + \sqrt{1 - e^2} \cos E \ell_2) (\partial E / \partial M)$$

$$(\partial y / \partial a) = (\cos E - e) m_1 + \sqrt{1 - e^2} \sin E m_2$$

$$(\partial y / \partial M) = a(-\sin E m_1 + \sqrt{1 - e^2} \cos E m_2) (\partial E / \partial M)$$

$$(\partial z / \partial a) = (\cos E - e) n_1 + \sqrt{1 - e^2} \sin E n_2$$

$$(\partial z / \partial M) = a(-\sin E n_1 + \sqrt{1 - e^2} \cos E n_2)(\partial E / \partial M)$$

$$\left(\frac{\partial \dot{x}}{\partial a}\right) = - \sqrt{\frac{\mu}{a^3}} \left[\frac{\sqrt{1 - e^2} \cos E \ell_2 - \sin E \ell_1}{(1 - e \cos E)} \right]$$

$$\left(\frac{\partial \dot{x}}{\partial M}\right) = - \sqrt{\frac{\mu}{a}} \left[\frac{\sqrt{1 - e^2} \sin E \ell_2 + (\cos E - e) \ell_1}{(1 - e \cos E)^2} \right] \left(\frac{\partial E}{\partial M}\right)$$

$$\left(\frac{\partial \dot{y}}{\partial a}\right) = - \sqrt{\frac{\mu}{a^3}} \left[\frac{\sqrt{1 - e^2} \cos E m_2 - \sin E m_1}{(1 - e \cos E)} \right]$$

$$\left(\frac{\partial \dot{y}}{\partial M}\right) = - \sqrt{\frac{\mu}{a}} \left[\frac{\sqrt{1 - e^2} \sin E m_2 + (\cos E - e) m_1}{(1 - e \cos E)^2} \right] \left(\frac{\partial E}{\partial M}\right)$$

$$\left(\frac{\partial \dot{z}}{\partial a}\right) = - \sqrt{\frac{\mu}{a^3}} \left[\frac{\sqrt{1 - e^2} \cos E n_2 - \sin E n_1}{(1 - e \cos E)} \right]$$

$$\left(\frac{\partial \dot{z}}{\partial M}\right) = - \sqrt{\frac{\mu}{a}} \left[\frac{\sqrt{1 - e^2} \sin E n_2 + (\cos E - e) n_1}{(1 - e \cos E)^2} \right] \left(\frac{\partial E}{\partial M}\right)$$

APPENDIX B
LAGRANGE PLANETARY EQUATIONS

APPENDIX B

LAGRANGE PLANETARY EQUATIONS

The Lagrange planetary equations are the following (ref. 5 or 6)

$$\begin{aligned}
 \frac{da}{dt} &= \frac{2}{na} \frac{\partial F}{\partial M} \\
 \frac{de}{dt} &= \frac{1-e^2}{na^2e} \frac{\partial F}{\partial M} - \frac{\sqrt{1-e^2}}{na^2e} \frac{\partial F}{\partial \omega} \\
 \frac{di}{dt} &= \frac{\cos i}{na^2 \sqrt{1-e^2} \sin i} \frac{\partial F}{\partial \omega} - \frac{1}{na^2 \sqrt{1-e^2} \sin i} \frac{\partial F}{\partial \Omega} \\
 \frac{d\Omega}{dt} &= \frac{1}{na^2 \sqrt{1-e^2} \sin i} \frac{\partial F}{\partial i} \\
 \frac{d\omega}{dt} &= - \frac{\cos i}{na^2 \sqrt{1-e^2} \sin i} \frac{\partial F}{\partial i} + \frac{\sqrt{1-e^2}}{na^2e} \frac{\partial F}{\partial e} \\
 \frac{dM}{dt} &= n - \frac{1-e^2}{na^2e} \frac{\partial F}{\partial e} - \frac{2}{na} \frac{\partial F}{\partial a}
 \end{aligned} \tag{B1}$$

The mean motion, n , and the disturbing function, F , are defined as

$$n = \sqrt{\frac{\mu}{a^3}} \tag{B2}$$

$$F = \frac{3}{2} \frac{\mu J_2 R^2}{a^3} \left(\frac{1}{3} - \frac{1}{2} \sin^2 i \right) (1-e^2)^{-3/2} \tag{B3}$$

If the time of periapsides passage is chosen as one of the independent variables, it is necessary to derive another equation as follows: Define (ref. 5)

$$\sigma = -nt_p$$

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then

$$\begin{aligned}\dot{\sigma} &= -\dot{n}t_p - n\dot{t}_p \\ &= -\frac{1-e^2}{na^2e} \frac{\partial F}{\partial e} - \frac{2}{na} \frac{\partial F}{\partial a}\end{aligned}$$

Therefore,

$$\begin{aligned}\dot{n}t_p &= \frac{1-e^2}{na^2e} \frac{\partial F}{\partial e} + \frac{2}{na} \frac{\partial F}{\partial a} - \dot{n}t_p \\ &= \frac{1-e^2}{na^2e} \frac{\partial F}{\partial e} + \frac{2}{na} \frac{\partial F}{\partial a} - \left(-\frac{3}{2} \frac{n}{a} \dot{a}\right) t_p \\ \frac{dt_p}{dt} &= \frac{1-e^2}{n^2a^2e} \frac{\partial F}{\partial e} + \frac{2}{na} \frac{\partial F}{\partial a} + \frac{3}{2} \frac{\dot{a}}{a} t_p\end{aligned}\quad (B4)$$

Substituting equation (B3) in equation (B4) gives, after some simplification

$$\frac{dt_p}{dt} = -\frac{3}{2} \frac{\mu J_2 R^2}{n^2 a^5} \left(1 - \frac{3}{2} \sin^2 i\right) (1-e^2)^{-3/2} \quad (B5)$$

Integrating gives

$$t_p = t_{p0} - \frac{3}{2} \frac{J_2 R^2}{p^2} (t - t_0) \left(1 - \frac{3}{2} \sin^2 i\right) \sqrt{1-e^2} \quad (B6)$$

where t_{p0} is a constant of integration equal to the time of periapsides passage at $t = t_0$ and $p = a(1 - e^2)$

Equation (B5) was obtained from equation 4 by setting $\dot{a} = 0$. This is not correct, for \dot{a} is periodic of small amplitude. As a result, equation (6) should give results which are in error, particularly for large values of time. Thus, the choice of t_p as one of the independent variables is not to be recommended. The use of the mean anomaly M as the sixth independent variable is to be preferred.

APPENDIX C
DERIVATION OF $\partial f / \partial M$

APPENDIX C

DERIVATION OF $\partial f / \partial M$

The derivation of the expression

$$\frac{\partial f}{\partial M} = \left(\frac{a}{r}\right)^2 \sqrt{1 - e^2} \quad (C1)$$

is as follows: From the expression

$$r \cos f = a(\cos E - e) \quad (C2)$$

$$\cos f = \frac{\cos E - e}{1 - e \cos E} \quad (C3)$$

since $r = a(1 - e \cos E)$. Recalling that $E = E(M, e)$, equation (C3) can be differentiated with respect to M as follows:

$$-\sin f \frac{\partial f}{\partial M} = \frac{-\sin E (\partial E / \partial M)}{(1 - e \cos E)} - \frac{(\cos E - e)(e \sin E)(\partial E / \partial M)}{(1 - e \cos E)^2} \quad (C4)$$

But

$$M = E - e \sin E$$

and

$$\frac{\partial E}{\partial M} = \frac{1}{(1 - e \cos E)} \quad (C5)$$

Substituting equation (C5) in equation (C4) gives

$$\begin{aligned} -\sin f \frac{\partial f}{\partial M} &= \frac{-\sin E}{(1 - e \cos E)^2} - \frac{(\cos E - e)(e \sin E)}{(1 - e \cos E)^3} \\ &= - \frac{\sin E}{(1 - e \cos E)^2} \left[1 + \frac{e \cos E - e^2}{(1 - e \cos E)} \right] \\ \frac{\partial f}{\partial M} &= \frac{\sin E}{\sin f (1 - e \cos E)^2} \left[\frac{1 - e \cos E + e \cos E - e^2}{(1 - e \cos E)} \right] \\ &= \frac{a \sin E (1 - e^2)}{r \sin f (1 - e \cos E)^2} \quad (C6) \end{aligned}$$

But

$$r \sin f = a \sqrt{1 - e^2} \sin E \quad (C7)$$

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Substituting equation (C7) in equation (C6) gives

$$\begin{aligned}\frac{\partial f}{\partial M} &= \frac{\sqrt{1 - e^2}}{(1 - e \cos E)^2} \\ &= \left(\frac{a}{r}\right)^2 \sqrt{1 - e^2}\end{aligned}\tag{C8}$$

$$\text{as } r = a(1 - e \cos E)$$

APPENDIX D

COMPUTATION OF PARTIAL DERIVATIVES OF THE ORBITAL ELEMENTS
AT TIME t WITH RESPECT TO THE ORBITAL ELEMENTS AT THE INITIAL TIME

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COMPUTATION OF PARTIAL DERIVATIVES OF THE ORBITAL ELEMENTS
AT TIME t WITH RESPECT TO THE ORBITAL ELEMENTS AT THE INITIAL TIME

The partial derivatives of the orbital elements at time t with respect to the orbital elements at the initial time t_0 are as follows:

$$(\partial a / \partial a_0) = 1$$

$$(\partial a / \partial e_0) = 0$$

$$(\partial a / \partial i_0) = 0$$

$$(\partial a / \partial \Omega_0) = 0$$

$$(\partial a / \partial \omega_0) = 0$$

$$(\partial a / \partial M_0) = 0$$

$$(\partial a / \partial t_{p_0}) = 0$$

$$(\partial e / \partial a_0) = 0$$

$$(\partial e / \partial e_0) = 1$$

$$(\partial e / \partial i_0) = 0$$

$$(\partial e / \partial \Omega_0) = 0$$

$$(\partial e / \partial \omega_0) = 0$$

$$(\partial e / \partial M_0) = 0$$

$$(\partial e / \partial t_{p_0}) = 0$$

$$(\partial i / \partial a_0) = 0$$

$$(\partial i / \partial e_0) = 0$$

$$(\partial i / \partial i_0) = 1$$

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$$(\partial i / \partial \Omega_0) = 0$$

$$(\partial i / \partial \omega_0) = 0$$

$$(\partial i / \partial M_0) = 0$$

$$(\partial i / \partial t_{p_0}) = 0$$

Define

$$\tilde{n} = n_0 N = \sqrt{\frac{\mu}{a_0^3}} \left[1 + \frac{3}{2} \frac{J_2 R^2}{p^2} \left(1 - \frac{3}{2} \sin^2 i \right) \sqrt{1 - e^2} \right]$$

where

$$n_0 = \sqrt{\mu / a_0^3}$$

$$N = 1 + \frac{3}{2} \frac{J_2 R^2}{p^2} \left(1 - \frac{3}{2} \sin^2 i \right) \sqrt{1 - e^2}$$

then

$$\begin{aligned} \frac{\partial \tilde{n}}{\partial a_0} &= \frac{\partial n_0}{\partial a_0} N + \frac{\partial N}{\partial a_0} n_0 \\ &= -\frac{3}{2} \frac{n_0}{a_0} N - \frac{3 J_2 R^2}{p^3} \left(1 - \frac{3}{2} \sin^2 i \right) (1 - e^2)^{3/2} \end{aligned}$$

$$\frac{\partial \tilde{n}}{\partial e_0} = n_0 \left[\frac{9}{2} \frac{J_2 R^2 e}{p^2 \sqrt{1 - e^2}} \left(1 - \frac{3}{2} \sin^2 i \right) \right]$$

$$\frac{\partial \tilde{n}}{\partial i_0} = n_0 \left[-\frac{9}{4} \frac{J_2 R^2}{p^2} \sqrt{1 - e^2} \sin(2i_0) \right]$$

Further, since $p = a(1 - e^2)$

$$(\partial p / \partial a_0) = 1 - e^2$$

$$(\partial p / \partial e_0) = -2ae$$

$$(\partial p / \partial i_0) = 0$$

Since

$$\Omega = \Omega_0 - \frac{3}{2} \frac{J_2 R^2}{p^2} \tilde{n} (\cos i) (t - t_0)$$

$$\frac{\partial \Omega}{\partial a_0} = -\frac{3}{2} \frac{J_2 R^2}{p^2} (\cos i) \left[\left(\frac{\partial \tilde{n}}{\partial p_0} \right) \left(\frac{\partial p}{\partial a_0} \right) - \left(\frac{\partial \tilde{n}}{\partial a_0} \right) \right] (t - t_0)$$

$$\frac{\partial \Omega}{\partial e_0} = \frac{3}{2} \frac{J_2 R^2}{p^2} (\cos i) \left[\left(\frac{\partial \tilde{n}}{\partial p} \right) \left(\frac{\partial p}{\partial e_0} \right) - \left(\frac{\partial \tilde{n}}{\partial e_0} \right) \right] (t - t_0)$$

$$\frac{\partial \Omega}{\partial i_0} = \frac{3}{2} \frac{J_2 R^2}{p^2} \left(\tilde{n} \sin i_0 - \cos i_0 \frac{\partial \tilde{n}}{\partial i_0} \right) (t - t_0)$$

$$(\partial \Omega / \partial \Omega_0) = 1$$

$$(\partial \Omega / \partial \omega_0) = 0$$

$$(\partial \Omega / \partial M_0) = 0$$

$$(\partial \Omega / \partial t_{p_0}) = 0$$

$$\omega = \omega_0 + \frac{3}{2} \frac{J_2 R^2 \tilde{n}}{p^2} \left(2 - \frac{5}{2} \sin^2 i_0 \right) (t - t_0)$$

$$\left(\frac{\partial \omega}{\partial a_0} \right) = \frac{3}{2} \frac{J_2 R^2}{p^2} \left(2 - \frac{5}{2} \sin i_0 \right) \left[\frac{\partial \tilde{n}}{\partial a_0} - \frac{2\tilde{n}}{p} \left(\frac{\partial p}{\partial a_0} \right) \right] (t - t_0)$$

$$\frac{\partial \omega}{\partial e_0} = \frac{3}{2} \frac{J_2 R^2}{p^2} \left(2 - \frac{5}{2} \sin^2 i_0 \right) \left[\frac{\partial n}{\partial e_0} - \frac{2n}{p} \left(\frac{\partial p}{\partial e_0} \right) \right] (t - t_0)$$

$$\left(\frac{\partial \omega}{\partial i_0} \right) = \frac{3}{2} \frac{J_2 R^2}{p^2} \left[\left(2 - \frac{5}{2} \sin^2 i_0 \right) \left(\frac{\partial \tilde{n}}{\partial i_0} \right) - \frac{5}{2} \tilde{n} \sin(2i_0) \right] (t - t_0)$$

$$(\partial \omega / \partial \Omega_0) = 0$$

$$(\partial \omega / \partial \omega_0) = 1$$

$$(\partial \omega / \partial M_0) = 0$$

$$(\partial \omega / \partial t_{p0}) = 0$$

$$M = M_0 + \tilde{n} (t - t_0)$$

$$\left(\frac{\partial M}{\partial a_0} \right) = \left(\frac{\partial \tilde{n}}{\partial a_0} \right) (t - t_0)$$

$$\left(\frac{\partial M}{\partial e_0} \right) = \left(\frac{\partial \tilde{n}}{\partial e_0} \right) (t - t_0)$$

$$\left(\frac{\partial M}{\partial i_0} \right) = \left(\frac{\partial \tilde{n}}{\partial i_0} \right) (t - t_0)$$

$$(\partial M / \partial \Omega_0) = 0$$

$$(\partial M / \partial \omega_0) = 0$$

$$(\partial M / \partial M_0) = 1$$

$$t_p = t_{p0} - \frac{3}{2} \frac{J_2 R^2}{p^2} \left(1 - \frac{3}{2} \sin^2 i_0 \right) \sqrt{1 - e_0^2} (t - t_0)$$

$$\left(\frac{\partial t_p}{\partial a_0} \right) = - \left(\frac{\partial N}{\partial a_0} \right) (t - t_0)$$

$$= \frac{3J_2 R^2}{p^3} \left(1 - \frac{3}{2} \sin^2 i_c \right) (1 - e_0^2)^{3/2} (t - t_0)$$

$$\begin{aligned}
 \left(\frac{\partial t_p}{\partial e_0} \right) &= - \left(\frac{\partial N}{\partial e_0} \right) (t - t_0) \\
 &= - \frac{9}{2} \frac{J_2 R^2 e_0}{p^2 \sqrt{1 - e_0^2}} \left(1 - \frac{3}{2} \sin^2 i_0 \right) (t - t_0)
 \end{aligned}$$

$$\begin{aligned}
 \left(\frac{\partial t_p}{\partial i_0} \right) &= - \left(\frac{\partial N}{\partial i_0} \right) (t - t_0) \\
 &= \frac{9}{4} \frac{J_2 R^2}{p^2} \sin(2i_0) \sqrt{1 - e_0^2} (t - t_0)
 \end{aligned}$$

$$(\partial t_p / \partial \Omega_0) = 0$$

$$(\partial t_p / \partial \omega_0) = 0$$

$$(\partial t_p / \partial t_{p0}) = 1$$

REFERENCES

1. Jezewski, Donald J.; and Rozendaal, Harvey L.: An Efficient Method for Calculating Optimal, Free-Space, N-Impulse Trajectories. MSC IN 67-FM-170, November 9, 1967.
2. Jezewski, Donald J.: A Method for Determining Optimal, Fixed-Time, N-Impulse Trajectories Between Arbitrarily Inclined Orbits. MSC IN 68-FM-159, July 2, 1968.
3. Battin, R. H.: Astronautical Guidance. McGraw-Hill Book Co., Inc., 1964.
4. Bond, Victor R.; and Faust, Nickolas L.: An Analytical Formulation of the State Transition Matrix Using a Universal Conic Variable. MSC IN 66-FM-99, September 19, 1966.
5. Roy, Archie E.: The Foundations of Astrodynamics. The Macmillan Company, 1965.
6. Escobal, Pedro Ramon: Methods of Orbit Determination. John Wiley & Sons, Inc., 1965.
7. Henry, Ellis W.: A Finite Differencing Method of Computing the State Transition Matrix for Any Type of Trajectory Model, Conic Through N-Body. MSC IN 68-FM-260, October 17, 1968.